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Second-order Cowley–Imai analogy in magnetogasdynamics

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The extended Cowley–Imai analogy is derived and employed to obtain explicit equations which allow transcription of gasdynamic perturbation solutions into magnetogasdynamic solutions. The transcription is written down to second order for axisymmetric super-Alfvénic flows of a perfect gas at arbitrary Mach numbers. Speed and field perturbations are shown to vanish in the Alfvénic limit for such solutions, although this is not a property of the exact solution. Van Dyke's supersonic-cone-flow solution is then transcribed and compared with the exact numerical solution over the range $1 < A_\infty \leqslant 20$, $1 < M_\infty \leqslant 20$ for a cone of 5° semi-apex angle, showing excellent agreement which improves with increasing field strength. The large-cone-angle behaviour of the solution is also quite good for the upstream state $A_\infty = M_\infty = 2{\cdot}0$.

1. Introduction

The aligned-field dissipationless flow of a perfect gas over a non-conducting slender body is of interest in magnetogasdynamics because of the insight into magnetogasdynamic phenomenology which may be gained by comparison with the corresponding gasdynamic solution. Unlike that in gasdynamics, the fundamental boundary-value problem here is not two-dimensional but is axisymmetric, since two-dimensional geometry would require that the current and vorticity loops present in the flow be open.

This paper employs the analogy of Cowley (1960) and Imai (1960) to transcribe solutions to this boundary-value problem from gasdynamics into solutions for magnetogasdynamics. In § 2 the analogy is extended in the manner of Grad (1960) and Peyret (1962) to include shocks and thermodynamic equations, and the range of its application to flows which are not homentropic, homenergic, or uniformly magnetized is discussed. The explicit transcription is worked out for the first two orders of axisymmetric potential flows with shock waves in § 3; general properties are also examined.

Finally, the Van Dyke (1952) cone-flow solution is transcribed in § 4. This solution is compared with the authors' (Bertram & Lynn 1972) numerical solution of the exact equations of supersonic super-Alfvénic cone flow. This is also compared with Bausset's (1963) first-order conical-flow perturbation solution, which is obtained by a similar transcription.

2. The extended Cowley–Imai analogy

The Lundquist equations for the magnetogasdynamic flow of a compressible dissipationless gas with aligned magnetic induction $\mathbf{B}$ and velocity $\mathbf{q}$ are as follows:

$$\nabla . (\rho \mathbf{q}) = 0, \tag{2.1}$$

$$\rho \mathbf{q} . \nabla \mathbf{q} + \nabla p = -\mathbf{B} \times (\nabla \times \mathbf{B})/\mu, \tag{2.2}$$

$$\nabla . [\rho \mathbf{q}(h + \tfrac{1}{2}q^2)] = 0, \tag{2.3}$$

$$\mathbf{B} = \alpha \rho \mathbf{q}, \quad \nabla . \mathbf{B} = 0, \qquad (2.4), (2.5)$$

with gas pressure p, density ρ, temperature T, specific entropy s, specific enthalpy h and specific internal energy e related by

$$p = p(\rho, s), \tag{2.6}$$

$$h = e + p/\rho = h(\rho, s), \tag{2.7}$$

$$T\,ds = de + p d(1/\rho), \tag{2.8}$$

where μ is the constant magnetic permeability and α is a scalar 'alignment constant', so-called because (2.1), (2.4) and (2.5) imply

$$\mathbf{q} . \nabla \alpha = 0, \tag{2.9}$$

i.e. $\alpha = \alpha(\psi)$ is a constant on a streamline given by $\psi =$ constant. Also, (2.3) is equivalent to the familiar Bernoulli equation

$$h + \tfrac{1}{2}q^2 = h_0(\psi) \tag{2.10}$$

or, when combined with (2.7) and (2.8),

$$s = s(\psi). \tag{2.11}$$

Cowley (1960) and Imai (1960), as well as Iur'ev (1960) and Hida (1961) (in different form), noted that, for homenergic ($h_0(\psi) =$ constant), homentropic ($s(\psi) =$ constant), uniformly magnetized flow ($\alpha(\psi) =$ constant) the substitutions

$$\mathbf{q}^* = (1 - 1/A^2)\,\mathbf{q}, \tag{2.12}$$

$$\rho^* = \rho/(1 - 1/A^2), \tag{2.13}$$

$$p^* = p + B^2/2\mu, \tag{2.14}$$

where the Alfvén number A is defined from the Alfvén speed b by

$$A^2 = q^2/b^2 = q^2/(B^2/\mu\rho) = \mu/\alpha^2\rho \tag{2.15}$$

after substitution of (2.4), reduce (2.1) and (2.2) to the corresponding gas-dynamic forms. Putting the Cowley–Imai variables from (2.12)–(2.14) into

(2.1)–(2.8), with varying α, h_0 and s, after some rearrangement results in

$$\nabla.(\rho^*\mathbf{q}^*) = 0, \tag{2.16}$$

$$\mathbf{q}^*.\nabla\left(\tfrac{1}{2}q^{*2}+\int a^{*2}\frac{d\rho^*}{\rho^*}\right) = 0, \tag{2.17}$$

$$\mathbf{q}^*\times(\nabla\times\mathbf{q}^*) = (1-1/A^2)\,[\nabla h_0 - T\nabla s-(b^2\nabla\alpha)/\alpha], \tag{2.18}$$

where $$a^{*2} = q^{*2}(A^2+M^2-1)/A^2M^2 = \partial p^*/\partial\rho^*, \tag{2.19}$$

the partial derivative being taken with α, h_0 and s held constant. Also, since $\rho^*\mathbf{q}^* = \rho\mathbf{q}$ the stream functions are identical: $\psi^* = \psi$.

We depart here from earlier derivations and introduce the thermodynamic variables

$$h^* = e^*+p^*/\rho^*, \tag{2.20}$$

$$T^*ds^* = de^*+p^*d(1/\rho^*), \tag{2.21}$$

where h^*, T^*, etc. are defined only by (2.20) and (2.21), by requiring that (2.18) should take the form of Crocco's relation

$$dh_0^* - T^*ds^* = (1-1/A^2)\,(dh_0 - T\,ds - b^2\,d\alpha/\alpha). \tag{2.22}$$

From (2.20)–(2.22) and (2.1)–(2.11) we have

$$de^* - dh_0^* = d[(1-1/A^2)\,(e-h_0)] \tag{2.23}$$

or, removing de^* from (2.23) by using (2.12), (2.13), (2.15) and (2.20),

$$dh_0^* = d(\tfrac{1}{2}q^{*2}+h^*). \tag{2.24}$$

At this point, all the flow and thermodynamic relations have been transcribed without specification of the fictitious thermodynamic quantities e^*, h^*, etc. in terms of corresponding actual flow variables, except that (2.22) requires (2.23) to hold.

Further restrictions must be placed on the fictitious thermodynamic variables if shock relations corresponding to (2.1)–(2.5) are to be written in terms of the Cowley–Imai variables:

$$[\rho^*q_n^*] = 0, \quad [\rho^*q_n^{*2}+p^*] = 0, \qquad (2.25), (2.26)$$

$$[\mathbf{q}_t^*] = 0, \quad [\tfrac{1}{2}q^{*2}+h^*] = 0, \quad [\alpha] = 0, \qquad (2.27), (2.28), (2.29)$$

where the 'jump' is $[X] = X_{\text{downstream}} - X_{\text{upstream}}$. While the other equations follow identically from previous definitions of the fictitious variables, (2.28) can be put into the form shown only by use of (2.23) and (2.24):

$$h^*+\tfrac{1}{2}q^{*2} = h_0^* = e^*+(1-1/A^2)\,(h_0-e), \tag{2.30}$$

within an additive constant. Thus $[h_0^*] = [e^*]-[(1-1/A^2)\,e+h_0/A^2]$ since $[h_0] = 0$, and we must have

$$e^* = (1-1/A^2)\,e+h_0/A^2 \tag{2.31}$$

and $$h_0^* = h_0(\psi). \tag{2.32}$$

That is, the transcription of shock relation (2.28) forces the choice of the Cowley–Imai variables e^*, h_0^*, and, through them, of h^*:

$$h^* = h_0^* - \tfrac{1}{2}q^{*2} = h_0 - (1 - 1/A^2)^2 (h_0 - h). \qquad (2.33)$$

Finally, s^* and T^* remain undefined except that once $s^*(\psi)$ is specified T^* must satisfy (2.22):

$$T^* = \{(1 - 1/A^2)\,(T\,ds/d\psi + (b^2/\alpha)\,d\alpha/d\psi) + (dh_0/d\psi)/A^2\}\,(d\psi/ds^*). \qquad (2.34)$$

The arbitrariness in the choice of $s^*(\psi)$ reflects the extra degree of freedom represented by α in the magnetogasdynamic flow. Note that neither of the obvious definitions, $s^* = (1 - 1/A^2)\,s$ and $T^* = (1 - 1/A^2)\,T$, will satisfy (2.34), because A^2 is not a function of ψ alone.

These same variables were written down by Peyret (1962) for the two-dimensional, uniformly magnetized, homenergic case, the latter two restrictions resulting from his arbitrary choice of $s^* = s$. Grad (1960) obtained the same transformation for the general case but selected h_0^* and h^* arbitrarily, without examination of the shock relations; h_0^* and e^* may be chosen in infinitely many ways and still satisfy the thermodynamic relations on the streamline, as shown by equation (2.30).

The Cowley–Imai analogy is complete when we add to the equations of state, (2.33) and (2.34), the equation for p^*:

$$p^* = p(\rho, s) + B^2/2\mu = p + \rho(h_0 - h)/A^2, \qquad (2.35)$$

where ρ and A^2 are given by

$$\left.\begin{aligned} \rho &= (1 - 1/A^2)\,\rho^* = \rho^*(1 + \alpha^2\rho^*/\mu)^{-1}, \\ A^2 &= 1 + \mu/\alpha^2\rho^*. \end{aligned}\right\} \qquad (2.36)$$

The full analogy is now defined by equations (2.16)–(2.22) and shock relations (2.25)–(2.28), along with the equations of state, (2.33)–(2.35). Because the latter change form under the transformation defined by (2.12)–(2.15) and (2.31)–(2.32), the analogous gas is 'fictitious'.

Interest here focuses on the use of the analogy to transcribe existing gas-dynamic solutions. For this use, Alfvénic flows ($A = 1$) must be excluded because ρ^* is infinite in this case; sub-Alfvénic flows must be excluded even if the transformation is redefined, with $(1/A^2 - 1)$ replacing $(1 - 1/A^2)$ in order to avoid negative ρ^*, since this leads to a change in sign of the first term of (2.26) and (2.27). Also, sub-Alfvénic hyperbolic flows require upstream-facing shocks, so that the region of influence of the body is unlike that of any existing gas-dynamic solution, a point noted by Grad (1960) but overlooked by Bausset (1963). Finally, studies of the exact equations for flow over a cone show that no hyperbolic sub-Alfvénic conical solution is possible because of singularities associated with the backward-facing characteristic from the nose of the body (Bertram & Lynn 1972). Thus the transcription is applied only to super-Alfvénic flows.

No limitation on Mach number has appeared in the derivation of the analogy, so it may be applied to subsonic or supersonic flows. Because of the unphysical nature of the equations of state, (2.33)–(2.35), only perturbation solutions will be transcribed.

3. Potential-flow transcription equations

To the first two orders, the solution to the perturbation equations for a slender body in a uniformly magnetized, homenergic, homentropic basic flow will also be homentropic, i.e. a potential flow, despite the presence of curved shock waves in the exact solution. For a potential flow the exact equations of motion may be written in terms of a perturbation potential by rearrangement of Imai's (1960) result by defining

$$\mathbf{q}^* = q_\infty^* \nabla(x+\phi^*), \tag{3.1}$$

where x is the axis of the unyawed slender body and the subscript ∞ refers to the undisturbed uniform flow. In Cartesian co-ordinates, with $\nabla h_0 = \nabla s = \nabla \alpha = 0$, (2.16)–(2.18) become, upon substitution of (3.1) (Imai 1960; Bertram 1969),

$$\begin{aligned}(1-M_\infty^{*2})\frac{\partial^2\phi^*}{\partial x^2}+\frac{\partial^2\phi^*}{\partial y^2}+\frac{\partial^2\phi^*}{\partial z^2} = M_\infty^{*2}\Bigg[&2\frac{\partial\phi^*}{\partial x}\frac{\partial^2\phi^*}{\partial x^2}-\frac{a^{*2}-a_\infty^{*2}}{a_\infty^{*2}}\nabla^2\phi^*+2\frac{\partial\phi^*}{\partial y}\frac{\partial^2\phi^*}{\partial x\, dy}\\ &+2\frac{\partial\phi^*}{\partial z}\frac{\partial^2\phi^*}{\partial x\,\partial z}+2\frac{\partial^2\phi^*}{\partial x^2}\left(\frac{\partial\phi^*}{\partial x}\right)^2+\frac{\partial^2\phi^*}{\partial y^2}\left(\frac{\partial\phi^*}{\partial y}\right)^2+\frac{\partial^2\phi^*}{\partial z^2}\left(\frac{\partial\phi^*}{\partial z}\right)^2\\ &+2\left(\frac{\partial\phi^*}{\partial x}\frac{\partial\phi^*}{\partial y}\frac{\partial^2\phi^*}{\partial x\,\partial y}+\frac{\partial\phi^*}{\partial y}\frac{\partial\phi^*}{\partial z}\frac{\partial^2\phi^*}{\partial y\,\partial z}+\frac{\partial\phi^*}{\partial z}\frac{\partial\phi^*}{\partial x}\frac{\partial^2\phi^*}{\partial z\,\partial x}\right)\Bigg],\end{aligned} \tag{3.2}$$

which is still exact and identical to the corresponding gasdynamic form (Van Dyke 1952). Expanding the pseudo-potential

$$\phi^* = \phi_1^* + \phi_2^* + \dots \tag{3.3}$$

results in, for the first two orders,

$$(1-M_\infty^{*2})\frac{\partial^2\phi_1^*}{\partial x^2}+\frac{\partial^2\phi_1^*}{\partial y^2}+\frac{\partial^2\phi_1^*}{\partial z^2} = 0, \tag{3.4}$$

$$\begin{aligned}(1-M_\infty^{*2})\frac{\partial^2\phi_2^*}{\partial x^2}+\frac{\partial^2\phi_2^*}{\partial y^2}+\frac{\partial^2\phi_2^*}{\partial z^2} = M_\infty^{*2}\Bigg[&2(N^*-1)\lambda_\infty^{*2}\frac{\partial\phi_1^*}{\partial x}\frac{\partial^2\phi_1^*}{\partial x^2}+2\frac{\partial\phi_1^*}{\partial y}\frac{\partial^2\phi_1^*}{\partial x\,\partial y}\\ &+2\frac{\partial\phi_1^*}{\partial z}\frac{\partial^2\phi_1^*}{\partial x\,\partial z}+\left(\frac{\partial\phi_1^*}{\partial y}\right)^2\frac{\partial^2\phi_1^*}{\partial y^2}+2\frac{\partial\phi_1^*}{\partial y}\frac{\partial^2\phi_1^*}{\partial y\,\partial z}+\left(\frac{\partial\phi_1^*}{\partial z}\right)^2\frac{\partial^2\phi_1^*}{\partial z^2}\Bigg].\end{aligned} \tag{3.5}$$

The triple products are retained on the right-hand side of (3.5) in anticipation of the axisymmetric 'gauge functions'. Equation (3.5) has been put into Van Dyke's (1952) form, with $\lambda_\infty^{*2} = M_\infty^{*2}-1$. The coefficients are

$$M_\infty^{*2} = A_\infty^2 M_\infty^2/(A_\infty^2+M_\infty^2-1) \tag{3.6}$$

and

$$N^* = M_\infty^2\frac{[(\gamma+1)(A_\infty^2-1)+3(M_\infty^2-1)]}{2(M_\infty^2-1)(A_\infty^2+M_\infty^2-1)}, \tag{3.7}$$

and (3.7) can be written in terms of an analogous polytropic index since the gasdynamic $N = \frac{1}{2}(\gamma+1)M_\infty^2/\lambda_\infty^2$:

$$\gamma^* = \left\{\frac{(A_\infty^2-1)[(\gamma+1)(A_\infty^2-1)+3(M_\infty^2-1)]}{A_\infty^2(A_\infty^2+M_\infty^2-1)}\right\}-1. \tag{3.8}$$

Note that both γ^* and M_∞^* reduce to their gasdynamic values in the vanishing field limit $A_\infty \to \infty$, while they become -1 and 1 respectively for $A_\infty \to 1$. Thus, the approach to Alfvénic transition for the real magnetogasdynamic flow corresponds to the approach of a Chaplygin–Kármán–Tsien gas (Sears 1954) to sonic transition for the fictitious gas, as far as the perturbation solution is concerned.

Since the shock relations are transcribed as (2.25)–(2.28), the fictitious gas will have the same weak-shock jumps as the magnetogasdynamic flow if the coefficients of the perturbation expansion of the equations of state are made to coincide by (3.8). Thus-weak shock solutions also transcribe, but it may be anticipated that, even near Alfvénic upstream flow no switch-on solutions will be obtained by transcription because no such gasdynamic solutions exist.

In summary, for an axisymmetric magnetogasdynamic flow in cylindrical co-ordinates (x, r, ϕ), the velocity is

$$\mathbf{q}(x,r) = u\hat{\mathbf{e}}_x + v\hat{\mathbf{e}}_r, \tag{3.9}$$

where $\hat{\mathbf{e}}_i$ is ith unit vector. Dividing through by a_∞ and expanding gives

$$u/a_\infty = M_\infty + u_1 + u_2 + \ldots, \tag{3.10}$$

$$v/a_\infty = v_1 + v_2 + \ldots. \tag{3.11}$$

The formulae from which the magnetogasdynamic velocity components are evaluated are now obtained by inverting the transformation equations (2.12)–(2.15), (2.31) and (2.32), and eliminating density by the Bernoulli equation (2.10) for a perfect gas.

$$\left.\begin{aligned} v_1 &= M_\infty v_1^*/q_\infty^*, \\ u_1 &= M_\infty \lambda_\infty^{*2}\left[\frac{u_1^*/q_\infty^* - \frac{1}{2}v_1^2/(A_\infty^2-1)}{M_\infty^2-1}\right], \end{aligned}\right\} \tag{3.12}$$

$$\left.\begin{aligned} v_2 &= M_\infty v_2^*/q_\infty^* + \rho_1 v_1/(A_\infty^2-1), \\ u_2 &= \frac{M_\infty\lambda_\infty^{*2}}{M_\infty^2-1}\left(\frac{u_2^*}{q_\infty^*} - \frac{v_1 v_2}{A_\infty^2-1} + \frac{\rho_1 u_1/M_\infty + R_2}{A_\infty^2-1}\right), \end{aligned}\right\} \tag{3.13}$$

$$\left.\begin{aligned} \rho_1 &= -M_\infty u_1 - \tfrac{1}{2}v_1^2, \\ R_2 &= \tfrac{1}{8}(2-\gamma)(v_1^2 + 4M_\infty u_1 v_1^2) - \tfrac{1}{2}[1-(2-\gamma)M_\infty^2]u_1^2, \\ \lambda_\infty^{*2} &= M_\infty^{*2} - 1 = \lambda_\infty^2 = (A_\infty^2-1)(M_\infty^2-1)/(A_\infty^2+M_\infty^2-1) \\ &= \cot^2(\text{characteristic angle}), \end{aligned}\right\} \tag{3.14}$$

where v_1^2 is the same order as u_1 on the body surface, according to the kinematic boundary condition

$$v_1/(M_\infty + u_1) = \epsilon\, dR_s/dx, \tag{3.15}$$

$$v_2/u_2 = \epsilon\, dR_s/dx, \tag{3.16}$$

with $\epsilon = R_{\max}/L \ll 1$ as the fineness ratio of the body given by

$$r_{\text{surface}} = \epsilon R_s(x), \quad R_s = O(1), \quad \text{for} \quad 0 \leqslant x \leqslant L. \tag{3.17}$$

Precision of the perturbation solution is significantly improved if the exact boundary conditions are satisfied at each order (Van Dyke 1957; Bertram 1969); apparently this is the result of ordering velocity components on the body surface. Note finally that the transcription equations (3.12)–(3.14) reduce to the gasdynamic solution in the limit $A_\infty \to \infty$.

In the Alfvénic limit, $A_\infty \to 1$, the fictitious gas becomes a Kármán–Chaplygin–Tsien gas and thus cannot reach the singular sonic-flow limit. Thus the gasdynamic solution v^*, u^* is finite, and repeated use of the relations $\lambda_\infty^2 = 0$ and $\lambda_\infty^2/(A_\infty^2 - 1) = 1 - 1/M_\infty^2$ in the limit shows that the speed perturbations

$$q_1 = u_1/M_\infty + v_1^2/2M_\infty^2, \quad q_2 = u_2/M_\infty + u_1^2/2M_\infty^2 + v_1 v_2/M_\infty^2$$

vanish in the limit $A_\infty \to 1$. So density, pressure and magnetic induction perturbations also vanish and only the direction of flow is perturbed by the body. This resembles the behaviour of the exact solution for the semi-infinite cone (Bertram & Lynn 1972), but occurs in the perturbation solution only at $A_\infty = 1$, when the shock should be a switch-off shock, rather than at a slightly greater value of A_∞, when the nose shock is a switch-on shock. This is purely a property of the transcription solution for the closed axisymmetric body, since a qualitative construction of the exact solution (Bertram 1969) includes expansions and varying flow on the body surface even when the nose shock is assumed to be a finite switch-on shock.

4. Transcription of the cone-flow solution

As a test case of the transcription process Van Dyke's (1952) solution for the gasdynamic supersonic flow over a semi-infinite cone of semi-apex angle θ_c is transcribed. The first-order solution

$$\left.\begin{aligned} v_1^*/q_\infty^* &= \lambda_\infty^* C_1(1/\xi^2 - 1)^{\frac{1}{2}}, \\ u_1^*/q_\infty^* &= -C_1 \operatorname{sech}^{-1}\xi + C_2, \end{aligned}\right\} \tag{4.1}$$

where $\xi = \lambda_\infty^* r/x = \lambda_\infty^* \tan\theta$, θ being the spherical co-ordinate polar angle, may be substituted into (3.12)–(3.14), giving

$$\left.\begin{aligned} v_1(\xi) &= M_\infty \lambda_\infty C_1(1/\xi^2 - 1)^{\frac{1}{2}}, \\ u_1(\xi) &= \frac{M_\infty \lambda_\infty^2}{M_\infty^2 - 1}\left[-C_1 \operatorname{sech}^{-1}\xi + C_2 - \frac{C_1^2 M_\infty^2}{2(A_\infty^2 - 1)}(1/\xi^2 - 1)\right]. \end{aligned}\right\} \tag{4.2}$$

To first order, the shock conditions require that the perturbations vanish on the backward-facing characteristic $\xi = 1$, so $C_2 = 0$. The exact boundary condition, equation (3.15), becomes

$$v_1(\delta)/[M_\infty + u_1(\delta)] = \tan\theta_c = \delta/\lambda_\infty,$$

where

$$\delta = \lambda_\infty \tan\theta_c = \xi_{\text{surface}},$$

so that

$$C_1 = \frac{-1 + \{1 + (2F_1(\delta)/\lambda_\infty^2)\,[((1+\lambda_\infty^2)/A_\infty^2)\,F_1(\delta)\,(1/\delta^2 - 1)]\}^{\frac{1}{2}}}{((1+\lambda_\infty^2)/A_\infty^2)\,F_1(\delta)\,(1/\delta^2 - 1)}, \tag{4.3}$$

where $F_1(\delta) = \delta^2/[(1-\delta^2)^{\frac{1}{2}} + \delta^2 \operatorname{sech}^{-1}\delta/(M_\infty^2 - 1)]$ is the gasdynamic C_1.

The second-order gasdynamic solution

$$\left.\begin{aligned} v_2^*/q_\infty^* &= \lambda_\infty^* C_3(1/\xi^2-1)^{\frac{1}{2}} + \lambda_\infty^* F_3(\xi), \\ u_2^*/q_\infty^* &= -C_3 \operatorname{sech}^{-1}\xi + C_4 + F_2(\xi), \end{aligned}\right\} \tag{4.4}$$

where
$$F_2(\xi) = C_1^2 M_\infty^{*2}\left[(\operatorname{sech}^{-1}\xi)^2 - (N^*-1)\frac{\operatorname{sech}^{-1}\xi}{1-\xi^2} - (N^*+1) - \tfrac{3}{4}\lambda_\infty^{*2} C_1 \frac{(1-\xi^2)^{\frac{1}{2}}}{\xi^2}\right],$$

$$F_3(\xi) = C_1^2 M_\infty^{*2}\left[-2\left(\frac{1}{\xi^2}-1\right)^{\frac{1}{2}}\operatorname{sech}^{-1}\xi + \frac{N^*+1}{\xi} + (N^*-1)\frac{\xi\operatorname{sech}^{-1}\xi}{(1-\xi^2)^{\frac{1}{2}}} + \frac{\lambda_\infty^{*2}C_1}{4}\frac{2+\xi^2}{\xi^3}(1-\xi^2)^{\frac{1}{2}}\right],$$

yields the magnetogasdynamic solution, with constants $C_4 = 0$ from the vanishing of perturbations on the characteristic, and

$$\begin{aligned} C_3 = \Bigg(&-F_3(\delta) - \frac{\rho_1(\delta)v_1(\delta)}{M_\infty\lambda_\infty(A_\infty^2-1)} + \frac{\delta}{(M_\infty^2-1)(A_\infty^2-1)} \\ &\times\left\{(A_\infty^2-1)F_2(\delta) - v_1(\delta)\left[\lambda_\infty M_\infty F_3(\delta) + \frac{\rho_1(\delta)v_1(\delta)}{A_\infty^2-1}\right] + \frac{\rho_1(\delta)u_1(\delta)}{M_\infty} + R_2(\delta)\right\}\Bigg) \\ &\times\left[\frac{\delta}{F_2(\delta)} + \frac{v_1(\delta)(1-\delta^2)^{\frac{1}{2}}\lambda_\infty M_\infty}{(M_\infty^2-1)(A_\infty^2-1)}\right]^{-1} \end{aligned} \tag{4.5}$$

from the exact second-order boundary condition, equation (3.16). Satisfying the shock relations to second order fixes the position of the shock wave at

$$\begin{aligned} \tan\beta &= \frac{1}{\lambda_\infty} + \frac{3}{2}\frac{M_\infty^{*4}N^{*2}}{\lambda_\infty}\tan^4\theta_c + \dots \\ &= \frac{1}{\lambda_\infty} + \frac{3}{8}\frac{A_\infty^4 M_\infty^3[(\gamma+1)(A_\infty^2-1) + 3(M_\infty^2-1)]^2}{(M_\infty^2-1)^{\frac{5}{2}}(A_\infty^2-1)^{\frac{1}{2}}(A_\infty^2+M_\infty^2-1)^{\frac{5}{2}}}\theta_c^4 + \dots \end{aligned} \tag{4.6}$$

by transcription of the result of Van Dyke (1952), obtained from a Lighthill stretching technique.

Bausset (1963) worked out the first-order solution, equation (4.2), and the shock-angle correction, equation (4.6), in a different form. His solution did not contain the v_1^2 term in u_1, since he ordered terms arbitrarily in powers of a general perturbation parameter. This led also to the appearance of the parameter in the perturbation terms implicitly, as well as explicitly in the coefficients. Finally, his application to sub-Alfvénic flows is incorrect because of the obtuse shock angle and the singularity at the backward characteristic off the nose, both of which were overlooked.

The precision of the solution, (4.2)–(4.6), is apparent in figures 1–3, where the results of numerical integration of the exact equations of motion (Bertram & Lynn 1972) are compared with the perturbation solution in the supersonic super-Alfvénic case. Figures 1 (*a*) and (*b*) compare surface velocity perturbation predictions, $\tilde{q}(\delta)$, for a semi-apex angle of 5° at fixed M_∞ and A_∞, respectively.

The second-order approximation is indistinguishable from the numerical ('exact') results over the entire range, while the first-order approximation improves in fit as the field strength increases. For the 5° cone with $A_\infty = M_\infty$, the divergence of the second-order approximation to the total pressure coefficient, $c_p = (p_\infty + B_\infty^2/2\mu - p_c - B_\infty^2/2\mu)/(\frac{1}{2}\rho_\infty q_\infty^2)$, is shown in figure 2. Again the strong-field fit is best, despite the inability of the perturbation solution to describe the switch-on shock.

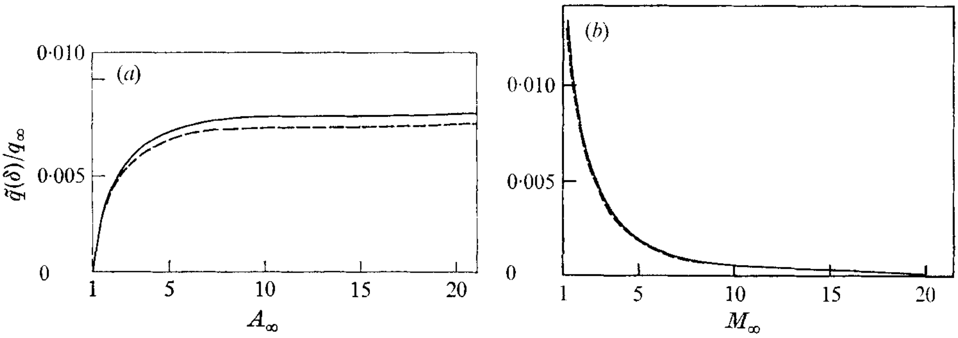

FIGURE 1. Perturbation solution, 5° cone. ——, exact; ------, first-order; $\gamma = \frac{5}{3}$.
(a) $M_\infty = 2{\cdot}0$; (b) $A_\infty = 2{\cdot}0$.

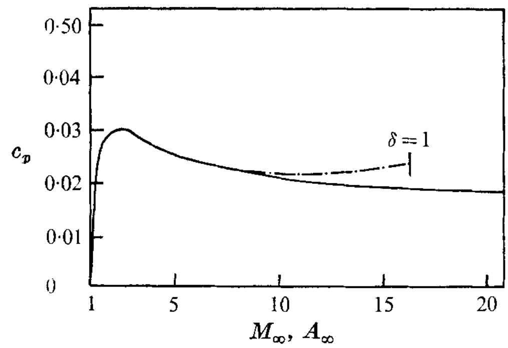

FIGURE 2. Perturbation solution, 5° cone. ——, exact; —·—, second-order;
$\gamma = \frac{5}{3}$; $A_\infty = M_\infty$.

In figure 3 the solution for surface velocity perturbations for all possible cone angles is plotted for strong- and weak-shock solutions when the fixed upstream state is $(A_\infty, M_\infty, \gamma) = (2{\cdot}0, 2{\cdot}0, \frac{5}{3})$. The first-order solution provides good approximations of surface velocity up to cone angles of 15°, while the second-order solution fits well until the perturbation parameter δ becomes unity, reproducing most of the weak-shock branch. However, unlike the first-order solution, the second-order solution breaks down in the flow while still giving good predictions at the surface. The breakdown begins when the prediction of velocity behind the shock becomes greater than the upstream velocity, implying that the shock is a rarefaction; as δ increases beyond this point more and more of the flow field becomes a rarefaction.

In summary, the perturbation solutions obtained from the Cowley–Imai analogy provide excellent approximations to the flow over a cone. The major limitations of the method are that it can only provide weak-shock solutions and that it cannot describe the switch-on shock solutions, even though the precision is greatest away from the shock in the strong-field case.

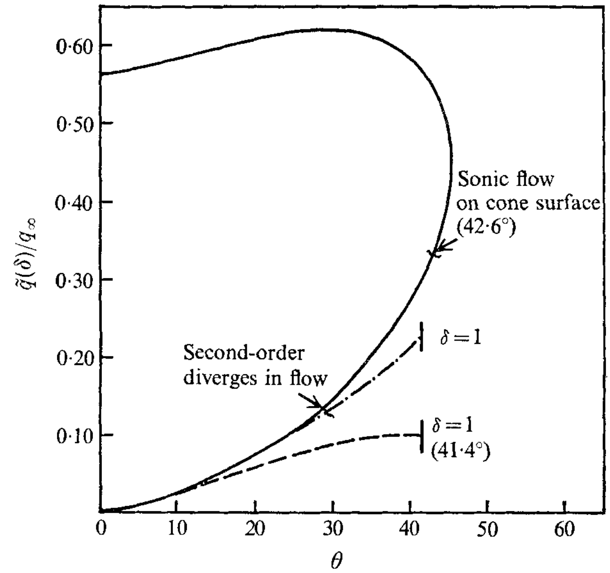

FIGURE 3. Perturbation solutions, large cones. ——, exact; – – –, first-order; —·—, second-order $\gamma = \frac{5}{3}$; $A_\infty = M_\infty = 2{\cdot}0$.

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